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Consumption and Savings Decisions Under Uncertainty : A Numerical Example*

INTRODUCTION

This paper shows how we might expect a rational consumer to decide upon his consumption plans in two periods when he is uncertain about income in the second period and can borrow or lend at a given competitive interest rate. Using relatively simple mathematics it is demonstrated how these plans are dependent upon the consumer's attitude towards risk and how they change under increasing uncertainty.

A substantial body of research on this question has been done in recent years by outstanding economists like Drèze, Modigliani, Sandmo, Fama, etc. [1] The main objective of their work is the development of a dynamic model which would explain simultaneously consumption, saving and investment decisions over several periods when future prospects are not known.

The model presented here undertakes an effort to illustrate some of the results with a simple numerical example. In order to do so three assumptions are made, the intertemporal preference function has a very specific functional form, income in the second period is uncertain but the probability distribution is given and one asset is available which offers a predetermined rate of return. The last restriction reduces the decision problem to a consumption and saving decision by excluding risky investment opportunities.

I. THE DESCRIPTION OF THE PROBLEM

Suppose a consumer faces the following decision problem. In the first period he has at his disposition an income Y_1 and he is allowed

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1. See the Bibliography.

to allocate it between immediate consumption C_1 and savings S_1 at an interest rate r. After the first period he receives an income Y_2, a random variable with probability distribution $f(Y_2)$, and his savings S_1 plus interest. Consumption C_2 in the second period is by definition equal to total income in this final period. It follows therefore that future consumption is dependent upon the random variable Y_2 and the consumption decision C_1. Consequently by deciding upon present consumption the consumer has already partly restricted his consumption choices in the final period.

The consumer expresses his preference for immediate and future consumption in a semi-cardinal utility function $U(C_1, C_2)$. It is reasonable to assume that the first partial derivatives of this utility function U_1 and U_2 are both positive.

The utility function specifies also the consumer's attitude towards risk. To make this clear, look at the following decision problem. Suppose the consumer has arrived by one way or another at a specific value for C_1 and faces the following prospects. He can choose between act 1 which gives him with certainty C_2 and act 2 a lottery which offers him $C_2 - h$ or $C_2 + h$, with equal probabilities. Table 1 summarizes this decision problem.

Act \ State	1	2
	$p = 1/2$	$1 - p = 1/2$
1	$U(C_1, C_2)$	$U(C_1, C_2)$
2	$U(C_1, C_2 - h)$	$U(C_1, C_2 + h)$

The expected utility for act 1 is $U(C_1, C_2)$ and for act 2 ;

$$\frac{1}{2} U(C_1, C_2 - h) + \frac{1}{2} U(C_1, C_2 + h).$$

A risk-averter will prefer act 1 to act 2. This implies :

$$U(C_1, C_2) > 1/2\, U(C_1, C_2 - h) + 1/2\, U(C_1, C_2 + h)$$
$$U(C_1, C_2) - U(C_1, C_2 - h) > U(C_1, C_2 + h) - U(C_1, C_2)$$

This is equivalent to saying that marginal utility is decreasing and consequently that the second partial derivative $U_{22} < 0$.

In a similar way we can show that a risk-lover will prefer act 2 to act 1 and will have $U_{22} > 0$. A consumer who is risk-neutral will be indifferent between both acts and will have $U_{22} = 0$.

It is clear that U_{22} gives us a measure for the degree of risk-aversion. The trouble is, however, that U_{22} is only definied up to a scale factor because of the semi-cardinal utility function. To derive a measure of risk-aversion we proceed as follows. If we reduce C_1 in the first act with an amount Θ there a situation arises where the risk-averter becomes indifferent between act 1 and act 2. This amount Θ is the risk-premium the decision maker is willing to pay to forego the risks of act 2. It is a measure of his risk-aversion.

We have now :

$$U(C_1, C_2 - \Theta) = \frac{1}{2} U(C_1, C_2 - h) + \frac{1}{2} U(C_1, C_2 + h)$$

Using Taylor expansions of the function on both sides, we obtain :

$$U(C_1, C_2) - \Theta U_2(C_1, C_2) + \frac{\Theta^2}{2} U_{22}(C_1, C_2) = U(C_1, C_2) + \frac{h^2}{2} U_{22}(C_1, C_2)$$

Because of $\Theta << h$, this reduces to :

$$-\Theta U_2(C_1, C^2) = \frac{h^2}{2} U_{22}(C_1, C_2)$$

$$\frac{2\Theta}{h^2} = \frac{U_{22}}{U_2}$$

The RHS gives us the Pratt-Arrow risk-aversion function. This measure is dependent upon C_1 and C_2.

2. See (1), (5).

II. THE GENERAL MODEL FOR RISK-AVERSION [3]

Data

Y_1 Income in the first period,

$f(Y_2)$ Probability distribution over Y_2, income in the second period $\bar{Y}_2 = E(Y_2) = \int Y_2 f(Y_2)\, d\, Y_2$

r Interest rate,

$U(C_1,C_2)$ Intertemporal utility function

$$U_1 > 0 \quad U_2 > 0 \quad U_{11} < 0 \quad U_{12} > 0 \quad U_{22} < 0$$

Decision variable : C_1

Model :

$Y_1 = C_1 + S_1$

$C_2 = Y_2 + S_1\ (1 + r)$

$C_2 = Y_2 + (Y_1 - C_1)\ (1 + r)$ Budget restriction

$E(U) = \int U(C_1,C_2)f(Y_2)\ dY_2$ Expected utility.

Maximization of expected utility with respect to C_1 subject to the budget restriction leads to :

$$\frac{dE(U)}{d} = 0 \quad \text{and} \quad \frac{d^2E(U)}{dC_1^2} < 0$$

or : $E(U_1 - (1 + r)U_2) = 0$

with the second order condition $E(U_{11} - 2(1+r)U_{12} + (1+r)^2U_{22}) < 0$

Solving the first order condition for C_1 gives the optimal consumption expenditures in the first periode as a function of income Y_1, the interest rate r and the probability distribution over income Y_2. It is possible to study the influence of a change in one of the parameters upon the equilibrium solution. This is postponed until we have derived solutions from a specific utility function.

3. See (6).

III. THE SPECIFIC MODEL

A. *Risk Aversion*

The utility function for this case is specified as follows :

$U(C_1,C_2) = C_1^{1/2} \,.\, C_2^{1/2}$

It is easy to verify that this function implies risk-aversion. The next hypothesis introduces the probability distribution $f(Y_2)$ which was taken a simple as possible :

$$f(Y_2) = \begin{cases} \dfrac{1}{\beta-\alpha} & \text{for } \alpha \leqslant Y_2 \leqslant \beta \\ 0 & \text{for } Y_2 < \alpha \,;\; Y_2 > \beta \end{cases}$$

$$E(Y_2) = \overline{Y}_2 = \mu = \frac{\beta+\alpha}{2}$$

$$\text{Var } (Y_2) = \Theta^2 = \frac{(\beta-\alpha)^2}{12}$$

It is possible to reformulate $f(Y_2)$.

$$f(Y_2) = \begin{cases} \dfrac{1}{\Theta\sqrt{12}} & \text{for } \mu - \Theta\sqrt{3} \leqslant Y_2 \leqslant \mu + \Theta\sqrt{3} \\ 0 & \text{for } Y_2 < \mu - \Theta\sqrt{3};\;\; Y_2 > \mu + \Theta\sqrt{3} \end{cases}$$

This particular continuous probability distribution assumes that income Y_2 has an equal probability of occurence between two extremes.

The effect of increased uncertainty upon the optimal immediate consumption can now be studied as a simple increase in the variance of $f(Y_2)$.

When we put $(Y_1 - C_1)\,(1+r) = A$, then the expected utility of the decision problem becomes :

$$E(U) = \int U(C_1,C_2) f(Y_2)\, dY_2$$

$$= \frac{C_1^{1/2}}{\beta-\alpha} \int_\alpha^\beta (Y_2 + A)^{1/2}\, dY_2$$

$$= \frac{2C_1^{1/2}}{3\,(\beta-\alpha)} \left((\beta + A)^{3/2} - (\alpha + A)^{3/2}\right)$$

The first order condition for a maximum brings us to the optimal solution for C_1.

$$\frac{dE(U)}{dC_1} = 0$$

$$(\beta+A)^{3/2}-(\alpha+A)^{3/2}= 3(1+r)C_1\,((\beta+A)^{1/2}-(\alpha+A)^{1/2})$$
$$((\beta+A)^{1/2}-(\alpha+A)^{1/2})\,(\beta+\alpha+2A$$
$$+((\beta+A)^{1/2}(\alpha+A)^{1/2}-3(1+r)C_1) = 0$$

Since $\beta \neq \alpha$

$$\beta + \alpha + 2A - 3(1 + r)C_1 + (\beta + A)^{1/2}\,(\alpha + A)^{1/2} = 0$$
$$\beta + \alpha + 2A - 3(1 + r)C_1 = -(\beta + A)^{1/2}\,(\alpha + A)^{1/2}$$

Squaring both sides leads to a quadratic equation in C_1. Solutions to this equation are given by :

$$C_1 = \frac{18(\mu + Y_1\,(1+r)) \pm 6\sqrt{(\mu + Y_1\,(1+r))^2 - 8\Theta^2}}{48\,(1 + r)}$$

Taking into account the second order condition gives the optimal C^*_1.

$$C^*_1 = \frac{3}{8}\left(\frac{\mu}{1+r} + Y_1\right) + \frac{1}{8}\sqrt{\left(\frac{\mu}{1+r} + Y_1\right) - \frac{8\Theta}{(1+r)^2}}$$

The *RHS* of the last equation contains a few well known elements. Indeed $\frac{\mu}{1+r}$ is the present value of expected income in the final period. From this we can construct the following expression :

$$Y_p = \frac{1}{2}\quad\frac{\mu}{1+r} + Y_1$$

Consequently Y_p is the average of actual income in the first period and the present value of expected income. This term is called permanent income because it plays a similar role as Friedman's permanent income concept.

It is now possible to rewrite the expression for the optimal C_1.

$$C^*_1 = \frac{3}{4}Y_p + \frac{1}{4}\sqrt{Y^2_p - \frac{2}{(1+r)^2}\,\Theta^2}$$

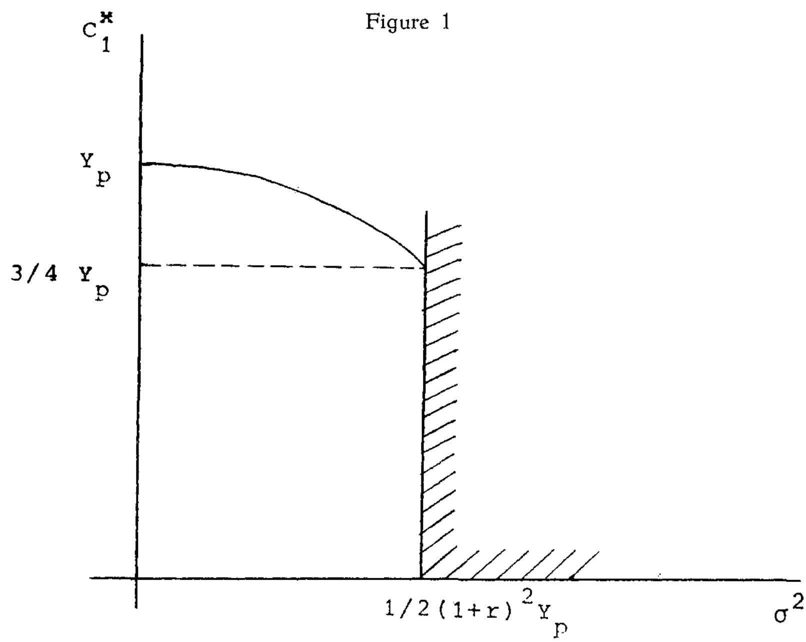
$$C^*_1 = \frac{3}{4}Y_p + \frac{1}{4}\sqrt{(Y_p - \frac{\sqrt{2}}{1+r}\Theta)\,(Y_p + \frac{\sqrt{2}}{1+r}\Theta)}$$

The optimal consumption expenditure appears as a weighted average of permanent income and the geometric average of two, for uncertainty corrected, symmetrical permanent incomes. From the fact that geometric averages are smaller than means it follows that a risk-averter will spend less than his permanent income on immediate consumption. This tendency will increase with increasing uncertainty.

When uncertainty diminishes consumption expenditures approach permanent income, which does not mean that no savings occur as will be shown below.

Figure 1 plots the functional relation between C^*_1 and uncertainty as expressed by the variance of the probability distribution.

Figure 1



It is clear from Figure 1 that increasing uncertainty about the future causes present consumption to decrease. However, beyond a certain level of uncertainty $\frac{1}{2}(1+r)^2 Y_p$ the mathematical solution loses its economic meaning. We are tempted to explain this as a discontinuity in the decision making pattern of the consumer, which occurs when perceived risk becomes too large.

The sensitivity of consumption to increasing uncertainty diminishes with increasing permanent income, this follows from the positivity of

$$\frac{\partial^2 C^*_1}{\partial Y_p \, \partial \Theta^2} = \frac{1}{4\,(1+r)^2} \, Y_p \left(Y^2_p - \frac{2\,\Theta^2}{(1+r)^2} \right)^{-3/2}$$

It is concluded therefore that people with higher permanent income care less about the uncertainty of future income.

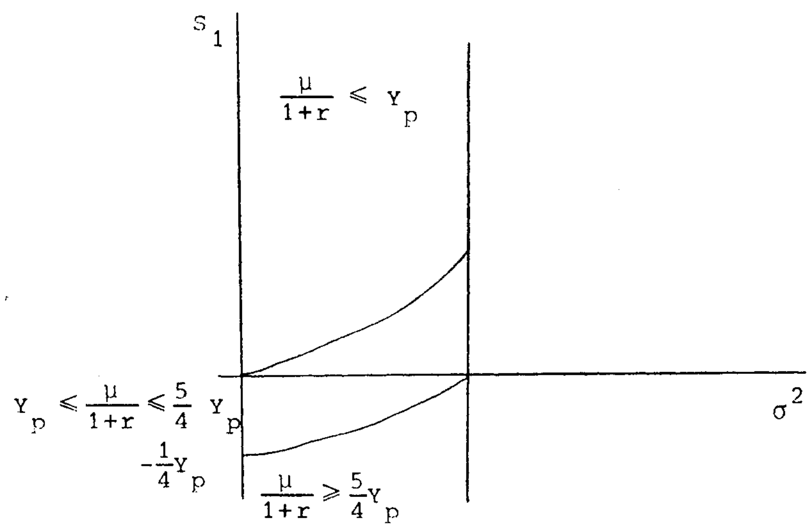
What does happen to savings ?

$$S_1 = Y_1 - C_1$$

$$S_1 = \frac{5}{4} Y_p - \frac{\mu}{1+r} - \frac{1}{4} \sqrt{Y^2_p - \frac{2}{(1+r)^2} \sigma^2}$$

Several factors influence savings. Permanent income, the degree of uncertainty, the present value of expected future income and the interest rate are all determinants of the saving decision. Figure 2 illustrates the functional relationship between savings and the degree of uncertainty. Whether savings or disavings occur depend upon permanent income and the present value of expected future income.

Figure 2



From Table 2 it is clear that under situation I all degrees of uncertainty give rise to negative savings, to the contrary situation III shows positive savings for all Θ^2. Under the conditions of situation II an unambiguous answer cannot be given. Whether savings or disavings are undertaken depends upon the degree of uncertainty.

I	$\dfrac{\mu}{1+r} > \dfrac{5}{3} Y_1$	For every Θ^2 dissaving
II	$Y_1 < \dfrac{\mu}{1+r} < \dfrac{5}{3} Y_1$	$0 < \Theta^2 < \Theta^2$ dissaving $\Theta^2 = \Theta^2_\star$ savings $= 0$ $\Theta^2_\star < \Theta^2$ saving
III	$\dfrac{\mu}{1+r} < Y_1$	For every Θ^2 saving

It seems reasonable to associate these different situations with different social groups in society. Younger people will be found in situation I, middle-aged people may be found in situation III. The savings behavior of different professional occupations may be explained by Table 2.

The conclusions advanced here are for risk-averters, we inquire now into two different cases, risk-neutrality and risk-loving.

B. *Rist-Neutrality*

The utility function is changed to take into account risk-neutrality.

$$U(C_1,C_2) = C_1.C_2$$

Similarly to the previous case, we calculate $E(U)$ and find Max $E(U)$. The optimal solution for C_1 is now very simple.

$$C^{\star}_1 = \frac{1}{2}\left(\frac{\mu}{1+r} + Y_1\right) = Y_p$$

Because of risk-neutrality, σ^2 does not show up any more in $C^{\star}_1$, and consumption expenditures are equal to permanent income.

C. *Risk-Loving*

After changing the utility function we go through the same calculations and derive C^*_1.

$$U(C_1 C_2) = C^2_1 . C^2_2$$

$$C^*_1 = \frac{3}{2} Y_p - \frac{1}{2} \sqrt{Y^2_p - \frac{2}{(1+r)^2} \Theta^2}$$

Here again permanent income plays an important role. Increased uncertainty however leads to higher consumption.

IV. CONCLUSIONS

Consistent with the literature we distinguished between three types of behavioral attitudes towards uncertainty, risk-aversion, risk-neutrality, and risk-loving. It is however possible to think of different names for the three cases.

In the first case we dealt with a prudent consumer who spends less to hedge against upcoming « disaster » (type 1), the last case described the behavior of a consumer who fully enjoys life and spends more to be sure that this is at least something which « disaster » cannot take away any more (type 2).

Figure 3

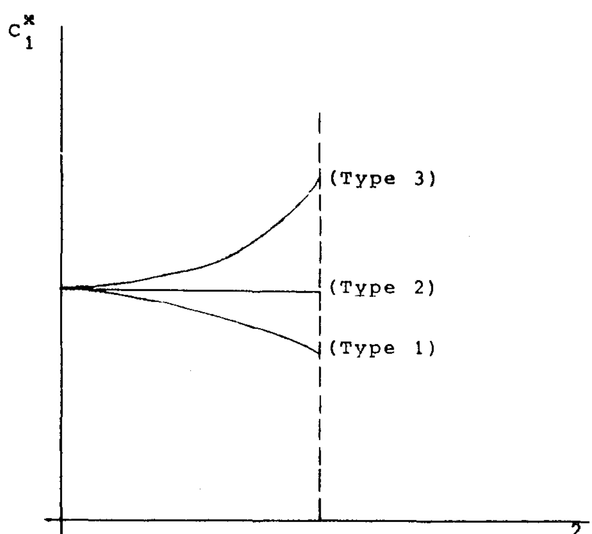


Figure 3 shows the three different types of behavior. They start from a common origin but spread out under influence of uncertainty.

The macro-economic consumption function is an aggregate of these micro-economic functions. The specific form of the macro relation is clearly dependent upon the distribution of the three behavioral modes in society.

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